578

VOLUME 80 SEPARATE No. 578

PROCEEDINGS

AMERICAN SOCIETY OF CIVIL ENGINEERS

DECEMBER, 1954

INTERACTION BETWEEN WELL AND AQUIFER

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HYDRAULICS DIVISION

{Discussion open until April 1, 1955}

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Printed in the United States of America

Headquarters of the Society
33 W. 39th St.
New York 18, N. Y.

PRICE $0.50 PER COPY

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This paper was published at 1745 S. State Street, Ann Arbor, Mich., by the American Society of Civil Engineers. Editorial and General Offices are at 33 West Thirty-ninth Street, New York 18, N. Y.

INTERACTION BETWEEN WELL AND AQUIFER

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SYNOPSIS

The piezometric head along a well is not constant as usually assumed, but it must decrease towards the discharging end of the well. An investigation is made to find the effect of this variation of piezometric head along the well on the flow in the aquifer. It is found that the formation loss to the discharging end of the well involves an additional term roughly equal to the velocity head at the discharging end of the well. The loss through the well screen is usually negligible. What is usually considered to be the screen loss is actually the additional loss mentioned above caused by the variation of piezometric head along the well. A solution is obtained for the axially symmetric flow from a confined uniform aquifer into a fully penetrating well.

INTRODUCTION

In the study of flow into wells, the assumption of a uniform distribution of piezometric head on the surface of the well screen is generally made. For a steady flow into a well fully penetrating a confined uniform aquifer, this assumption leads to a two-dimensional radial-flow solution which has been widely used in practice. This well known solution is

$$H - H_{od} = H - H_o = \frac{Q_d}{2\pi kL} \log_e\left(\frac{2r}{D}\right) \qquad (1)$$

in which H is the piezometric head in the aquifer, Q is the rate of discharge in the well, k is the coefficient of permeability of the soil (dimensionally, length per unit time), L is the thickness of the aquifer, D_o is the outside diameter of the well, and r is the horizontal distance from the center of the well (see Fig. 1). The subscripts o and d are used to signify values at the outside surface of the well screen and at the discharging end of the well respectively. (The letter symbols used in this paper have been collected in Appendix II for convenience of reference). However, by energy or momentum consideration, the piezometric head h in the well must decrease towards the discharging end of the well. It follows that the piezometric head H_o in the soil just outside the well cannot be uniformly distributed along the screen. The purpose of the present investigation is to find, through a theoretical analysis based on well established principles, the effect of this variation of piezometric head along the well on the flow in the aquifer. A steady flow from a confined uniform aquifer into a fully penetrating well is used for demonstration.

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The Flow in the Well

In the flow in the well, the piezometric head h, the rate of discharge Q and the rate of flow added to the well through the screen dQ/dz, vary with z along the well. The vertical distance z is measured from the bottom of the well as shown in Fig. 1. The equations governing the flow are the momentum equation,

$$h - h_d = \frac{Q_d^2}{gA^2} - \frac{Q^2}{gA^2} \tag{2}$$

the equation of continuity,

$$Q = Q_d - \int_z^L \frac{dQ}{dz}\, dz \tag{3}$$

and the orifice equation for the screen openings,

$$\frac{dQ}{dz} = C_c A_p \pi D \sqrt{2g(H_o - h)} = C_s \pi D \sqrt{2g(H_o - h)} \tag{4}$$

in which A is the inside cross-sectional area of the well, g is the gravitational acceleration, C_c is the coefficient of discharge for the screen openings, A_p is the screen opening area per unit area of screen surface, $C_s = C_c A_p$ is the screen coefficient, and D is the inside diameter of the well. In these equations, the quantities h, Q, dQ/dz and H_o vary with z. In the momentum equation, the friction on the screen-wall has been neglected. The screen coefficient C_s in Eq. 4 can be determined experimentally. The validity of Eqs. 2 and 4 has been demonstrated experimentally by Petersen, Rohwer and Albertson.[2] The principle of conservation of energy is not used here since the loss of energy, mainly due to the impact between the incoming water and the water already flowing in the well, is unknown a priori. The loss of energy in the flow, however, can be determined from Eq. 2.

The flow in the aquifer must be so adjusted that these three equations are satisfied. Substitute into Eq. 2 the value of Q from Eq. 3 and the value of h from Eq. 4,

$$H_o - h_d - \frac{1}{2g(C_s \pi D)^2}\left(\frac{dQ}{dz}\right)^2 = \frac{2Q_d}{gA^2}\int_z^L \frac{dQ}{dz}\, dz - \frac{1}{gA^2}\left(\int_z^L \frac{dQ}{dz}\, dz\right)^2 \tag{5}$$

Eq. 5 is the condition that the values of H_o and dQ/dz must satisfy at every point z along the screen.

2. Petersen, Rohwer and Albertson, "Effect of Well Screen on Flow into Wells", Proc. ASCE Vol. 79, Separate No. 365, Dec. 1953.

The Flow in the Aquifer

In this investigation, attention is confined to cases with laminar flow throughout the aquifer. For these cases, the flow throughout the aquifer follows Darcy's Law for seepage:

$$q_r = -k\frac{\partial H}{\partial r} \quad ; \quad q_z = -k\frac{\partial H}{\partial z} \tag{6}$$

in which q_r and q_z are the volume of flow per unit time per unit cross-sectional area of soil in the r- and z-direction respectively. For continuity in this axial-symmetric flow, the distribution of the piezometric head H in the aquifer must satisfy the following equation:[3]

$$\frac{\partial^2 H}{\partial z^2} + \frac{1}{r}\cdot\frac{\partial H}{\partial r} + \frac{\partial^2 H}{\partial r^2} = 0 \tag{7}$$

A solution of Eq. 7 for the steady flow in a uniform aquifer, satisfying Eq. 5 at the screen of a fully penetrating well, can be obtained for cases satisfying the following condition:

$$\frac{Q_d k D_e}{gA^2} \ll \frac{K_o\left(\frac{\pi D_e}{2L}\right)}{K_1\left(\frac{\pi D_e}{2L}\right)} \tag{8}$$

This condition is generally satisfied in practical cases. The solution is (see Appendix I)

$$H = H_{od} + \frac{Q_d}{2\pi kL}\log_e\left(\frac{2r}{D_e}\right) + \frac{8}{\pi^2}\cdot\frac{Q_d^2}{2gA^2}\sum_{n=1}^{\infty}\frac{1}{n^2(1+N_n)}$$
$$- \frac{8}{\pi^2}\cdot\frac{Q_d^2}{2gA^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2 K_o\left(\frac{n\pi D_e}{2L}\right)(1+N_n)}\cdot K_o\left(\frac{n\pi r}{L}\right)\cdot\cos\left(\frac{n\pi z}{L}\right) \tag{9}$$

in which

$$N_n = \left(\frac{\pi}{4}\right)^2\cdot\frac{Q_d k D_e}{gA^2}\cdot\left(\frac{D}{C_s L}\right)^2\cdot n\,K_1\left(\frac{n\pi D_e}{2L}\right)\Big/K_o\left(\frac{n\pi D_e}{2L}\right) \tag{9a}$$

3. Muskat, M., "Flow of Homogeneous Fluids Through Porous Media," McGraw Hill Co., 1937, p. 141.

and, for practical purposes,

$$\sum_{n=1}^{\infty} \frac{1}{n^2(1+N_n)} \simeq \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \qquad (9b)$$

since the contribution of N_n to the sum of the series is negligible in practical cases. In these equations, K_0 and K_1 are the modified Bessel Functions of the second kind of zeroth and first order respectively. Both K_0 and K_1 decrease with an increase of the argument in a manner similar to the exponential functions. Eq. 9 gives the three-dimensional distribution of piezometric head in the aquifer. The flow along the top of the aquifer experiences a greater loss of head than that along the bottom. The difference can be found from Eq. 9 to be $Q_d^2/(gA^2)$ and is shown schematically in Fig. 1.

The distribution of flow in the aquifer can be obtained from Eq. 9 according to Eq. 6. The distribution of inflow to the well, dQ/dz, can be shown to be

$$\frac{dQ}{dz} = \pi D_o k\left(\frac{\partial H}{\partial r}_o\right) = \frac{Q_d}{L}\left[1 - 4\,\frac{Q_d k D_o}{gA^2}\sum_{n=1}^{\infty}\frac{(-1)^n K_1\left(\frac{n\pi D_o}{2L}\right)\cdot\cos\left(\frac{n\pi z}{L}\right)}{n(1+N_n)\,K_o\left(\frac{n\pi D_o}{2L}\right)}\right] \qquad (10)$$

Since the last term in the parenthesis does not amount to more than 10^{-2} in practical cases,

$$\frac{dQ}{dz} \simeq \frac{Q_d}{L} \qquad (11)$$

This shows that the velocity distribution is practically the same as the radial-flow approximation although the distribution of piezometric head near the well is quite different from the approximate solution (Eq. 1). The explanation for this is that, due to the high resistance offered to the flow by the aquifer, a slight change of velocity distribution would cause an appreciable change of piezometric head distribution.

Practical Significance

Although the actual distribution of the piezometric head or of the velocity of flow is of little practical interest, Eq. 9 yields interesting results with regard to the formation loss and the loss across the screen.

Formation loss to discharging end of well

The head loss in the aquifer from a point at distance r to the discharging end of the well is $(H-H_{od})$. As r increases, the last term in Eq. 9 rapidly diminishes. Thus for reasonably large value of r (say 2L or larger), the value of $(H-H_{od})$ is

$$H - H_{od} \simeq \frac{Q_d}{2\pi kL}\log_e\left(\frac{2r}{D_o}\right) + \frac{4}{3}\cdot\frac{Q_d^2}{2gA^2}, \quad (r \geq 2L) \qquad (12)$$

It is thus shown that the formation loss to the discharging end of the well is greater than that obtained from the approximate radial-flow solution (Eq. 1) by an amount equal to 4/3 times the velocity head of the flow in the well at the discharging end (see Fig. 1).

In this analysis, only cases with laminar flow throughout the aquifer are considered. For cases with turbulent flow near the well, there will be an additional loss of head.

Head loss across the screen

To find the difference in piezometric head across the screen, Eq. 4 may be reduced to read

$$H_o - h = \frac{Q_d^2}{2gA^2} \cdot \left(\frac{D}{4C_s L}\right)^2 \left(\frac{dQ}{dz}\right)^2 \Big/ \left(\frac{Q_d}{L}\right)^2 \qquad (13)$$

Substitute Eq. 11 into Eq. 13,

$$H_o - h \simeq \left(\frac{D}{4C_s L}\right)^2 \cdot \frac{Q_d^2}{2gA^2} \qquad (13a)$$

It can be seen from this equation that, with reasonable values for the screen coefficient[2], the piezometric head difference across any point of the screen amounts to only a very small fraction of the velocity head of the well discharge.

Thus the total loss of piezometric head to the discharging end of the well is

$$H - h_d \simeq \frac{Q_d}{2\pi kL} \log_e\left(\frac{2r}{D_o}\right) + \frac{4}{3} \cdot \frac{Q_d^2}{2gA^2} + \left(\frac{D}{4C_s L}\right)^2 \frac{Q_d^2}{2gA^2}, \quad (r \geq 2L) \qquad (14)$$

This is the sum of the formation loss along the top of the aquifer and the loss through the screen. The same loss of piezometric head can be obtained along any other streamline. For example, the formation loss along the bottom of the aquifer to the bottom of the well is less than that along the top by an amount equal to $Q_d^2/(gA^2)$ (from Eq. 9, see Fig. 1), but the loss of piezometric head inside the well from the bottom to the top is equal to $Q_d^2/(gA^2)$ (from Eq. 2), thus making the total loss to the discharging end of the well the same as that of the flow along the top of the aquifer.

When no turbulent flow in the aquifer is involved, the data obtained from well tests are usually expressed as

$$H - h_d = aQ_d + bQ_d^2 \qquad (15)$$

where a and b are constants for a given well and given r. The term aQ_d is usually referred to as the formation loss while the term bQ_d^2 is considered to be the screen loss. By comparing Eqs. 14 and 15, the term aQ_d does not represent the entire formation loss, and the term bQ_d^2 is mainly that part of the loss caused by the variation of piezometric head along the well.

CONCLUSIONS

Although the analysis presented in this paper is confined to the case of a fully penetrating well in a confined aquifer, the following conclusions are believed to be applicable to wells in general.

1) With laminar flow throughout the aquifer, the formation loss to the discharging end of the well consists of two parts. One part is of the same order of magnitude as the velocity head in the well at the discharging end. (For a fully penetrating well, this part is equal to 4/3 times the velocity head). The other part is equal to that obtained with the assumption of uniform piezometric head at the well surface.

2) The velocity distribution in the aquifer obtained by assuming uniform piezometric head at the well is accurate enough for practical purposes.

3) Since the velocity distribution obtained by assuming uniform piezometric head at the well is accurate enough for practical purposes, Eq. 13 may be used for finding the difference of piezometric head across the screen with the approximate value of dQ/dz thus obtained. This head loss across the screen is negligible for screens in reasonably fair condition. What is generally considered to be the screen loss in Eq. 15 is actually the additional loss caused by the variation of piezometric head along the screen.

ACKNOWLEDGMENT

The author is indebted to Dr. Shih-Kung Kao of The Johns Hopkins University for his helpful suggestions in connection with the analysis of this problem.

APPENDIX I - SOLUTION OF EQ. 7 FOR WELL SHOWN IN FIG. 1

Let $\phi = kHL/Q_d$, $x = r/L$, $y = z/L$, $\delta = D_o/2L$, $\alpha = Q_d k D_o/(g A^2)$ and $\beta = (D/C_s L)^2$. Eq. 7 can then be rewritten as

$$\frac{\partial^2 \phi}{\partial y^2} + \frac{1}{x}\frac{\partial \phi}{\partial x} + \frac{\partial^2 \phi}{\partial x^2} = 0 \qquad (16)$$

For the aquifer and well shown in Fig. 1, the boundary conditions to be satisfied are

1) $\frac{\partial \phi}{\partial y} = 0$ at $y = 0$, since q_z must vanish there;

2) $\frac{\partial \phi}{\partial y} = 0$ at $y = 1$, since q_z must vanish there;

3) $\frac{\partial \phi}{\partial x} \rightarrow \frac{1}{2\pi x}$ as $x \rightarrow \infty$, since at large distance from the well the flow is essentially radial;

4) $\frac{\partial \phi}{\partial y} \rightarrow 0$ as $x \rightarrow \infty$, for the same reason;

5) $\phi = \phi_{od}$ at $x = \delta$ and $y = 1$, by definition; and

6) For any point along the well screen, Eq. 5 must be satisifed.

Since by continuity of flow across the screen,

$$\frac{dQ}{dz} = \pi D_o k \left(\frac{\partial H}{\partial r}\right)_o \tag{17}$$

Eq. 5 may be rewritten as

$$\phi_o - \frac{k h_d L}{Q_d} - \left(\frac{\pi}{4}\right)^2 \alpha\beta\delta \left(\frac{\partial \phi}{\partial x}\right)_o^2 = 2\pi\alpha \int_y^1 \left(\frac{\partial \phi}{\partial x}\right)_o dy - 2\pi^2 \alpha\delta \left[\int_y^1 \left(\frac{\partial \phi}{\partial x}\right)_o dy\right]^2 \tag{18}$$

Since at y = 1, the two terms involving an integral vanish, and $\phi_o = \phi_{od}$ and $\left(\frac{\partial \phi}{\partial x}\right)_o = \left(\frac{\partial \phi}{\partial x}\right)_{od}$ by definition,

$$\frac{k h_d L}{Q_d} = \phi_{od} - \left(\frac{\pi}{4}\right)^2 \alpha\beta\delta \left(\frac{\partial \phi}{\partial x}\right)_{od}^2 \tag{19}$$

Thus B.C. (6) is that, for any value of y at $x = \delta$, the following condition must be satisfied:

$$\phi_o - \phi_{od} + \left(\frac{\pi}{4}\right)^2 \alpha\beta\delta \left[\left(\frac{\partial \phi}{\partial x}\right)_{od}^2 - \left(\frac{\partial \phi}{\partial x}\right)_o^2\right] = 2\pi\alpha \int_y^1 \left(\frac{\partial \phi}{\partial x}\right)_o dy - 2\pi^2 \alpha\delta \left[\int_y^1 \left(\frac{\partial \phi}{\partial x}\right)_o dy\right]^2 \tag{20}$$

Let

$$\phi = F(x) \cdot G(y) \tag{21}$$

Eq. 16 can then be written as

$$\frac{F'}{rF} + \frac{F''}{F} = -\frac{G''}{G} = c^2 \tag{22}$$

where c must be a constant. Furthermore, to satisfy B.C. (1) and B.C. (2) simultaneously, no solution is possible if c is an imaginary number. Therefore c must be a real constant. Here the primes and double-primes signify the first and second derivatives of the functions respectively. Two cases must be considered separately, with c = 0 and with $c \neq 0$ respectively.

1) When c = 0:
When c = 0, G'' = 0. Then G' = A_0 where A_0 is a constant. In order to satisfy B.C. (1), (2) or (4), A_0 must be zero. Therefore

$$G = B_0 \tag{23}$$

where B_0 is a constant. With c = 0, Eq. 22 gives

$$F = A_1 + B_1 \log_e x \tag{24}$$

where A_1 and B_1 are constants. Thus, with c = 0,

$$\phi = F \cdot G = a_0 + b_0 \log_e x \tag{25}$$

where a_0 and b_0 are constants.

2) When c ≠ 0:
When c ≠ 0, Eq. 22 gives

$$G = A_2 \cos(cy) + B_2 \sin(cy) \tag{26}$$

and

$$G' = -cA_2 \sin(cy) + cB_2 \cos(cy) \tag{27}$$

To satisfy B.C. (1), G' must vanish at y = 0. Therefore $cB_2 = 0$, or $B_2 = 0$ since c ≠ 0. To satisfy B.C. (2), G' must vanish at y = 1. Therefore $cA_2 \cdot \sin(c) = 0$. Since c ≠ 0, and with vanishing A_2 the solution would be trivial, we must have sin(c) = 0. Therefore $c = n\pi$, where n = 1,2,3,....., and

$$G = a_n \cos(n\pi y), \quad (n = 1,2,3,\cdots\cdots) \tag{28}$$

where a_n is a constant for each value of n. With $c = n\pi$, Eq. 22 also gives

$$F = A_3 I_o(n\pi x) + B_3 K_o(n\pi x) \tag{29}$$

where I_0 and K_0 are the modified Bessel Functions of zeroth order of the first and the second kind respectively. The value of I_0 increases while the value of K_0 decreases with an increase in the value of the argument in a manner similar to the exponential functions. To satisfy B.C. (4), FG' must approach zero as x increases. Since G' does not vanish identically, F must approach zero as x increases. Therefore A_3 must be zero. Thus, when $c = n\pi$,

$$\phi = F \cdot G = a_n K_o(n\pi x) \cdot \cos(n\pi y), \quad (n = 1,2,3,\cdots\cdots) \tag{30}$$

The constant B_3 in Eq. 29 has been absorbed into a_n.

Since both Eq. 25 and Eq. 30 are solutions of Eq. 16 which is linear, the general solution of Eq. 16 is therefore

$$\phi = a_o + b_o \log_e x + \sum_{n=1}^{\infty} a_n K_o(n\pi x) \cdot \cos(n\pi y) \qquad (31)$$

The constant b_o can be determined from B.C. (3): Using the identity

$$\frac{d}{dx} K_o(n\pi x) = -n\pi \cdot K_1(n\pi x) \qquad (32)$$

we have

$$\frac{\partial \phi}{\partial x} = \frac{b_o}{x} - \sum_{n=1}^{\infty} n\pi a_n K_1(n\pi x) \cdot \cos(n\pi y) \qquad (33)$$

B.C. (3) is satisfied with $b_o = 1/(2\pi)$, since $K_1(n\pi x)$, the modified Bessel Function of first order of the second kind, vanishes at large values of x. The constant a_o can be determined from B.C. (5): At $x=\delta$ and $y = 1$, Eq. 31 and B.C. (5) gives

$$\phi_{od} = a_o + \frac{1}{2\pi} \log_e \delta + \sum_{n=1}^{\infty} (-1)^n a_n K_o(n\pi\delta) \qquad (34)$$

Therefore

$$\phi = \phi_{od} + \frac{1}{2\pi} \log_e \left(\frac{x}{\delta}\right) - \sum_{n=1}^{\infty} (-1)^n a_n K_o(n\pi\delta) + \sum_{n=1}^{\infty} a_n K_o(n\pi x) \cdot \cos(n\pi y) \qquad (35)$$

This solution of Eq. 16 satisfies the first five boundary conditions. The coefficients a_n must be such that B.C. (6) is also satisfied.

Eq. 35 indicates that the solution consists of the solution for two-dimensional radial flow (Eq. 1) with additional perturbation terms under the summation signs due to the non-uniform distribution of piezometric head along the well screen. To linearize Eq. 20 (B.C. 6), cross-products of the small perturbation terms are neglected. For example, we have from Eq. 33 the value of $\frac{\partial \phi}{\partial x}$ and we use in Eq. 20

$$\left(\frac{\partial \phi}{\partial x}\right)^2_{od} = \left(\frac{1}{2\pi\delta}\right)^2 - 2\left(\frac{1}{2\pi\delta}\right)\sum_{n=1}^{\infty}(-1)^n n\pi\, a_n K_1(n\pi\delta) \quad (36)$$

In this manner, Eq. 20 may be reduced to the following form:

$$\sum_{n=1}^{\infty} a_n K_o(n\pi\delta)\cdot\cos(n\pi y) - \sum_{n=1}^{\infty}(-1)^n a_n K_o(n\pi\delta)$$

$$+\left(\frac{\pi}{4}\right)^2\alpha\beta\sum_{n=1}^{\infty} n\, a_n K_1(n\pi\delta)\cdot\cos(n\pi y) - \left(\frac{\pi}{4}\right)^2\alpha\beta\sum_{n=1}^{\infty}(-1)^n n\, a_n K_1(n\pi\delta) \quad (37)$$

$$-2\pi\alpha\sum_{n=1}^{\infty} a_n K_1(n\pi\delta)\cdot y\cdot\sin(n\pi y) - \frac{\alpha}{2\delta}(1-y^2) = 0$$

The values of the constants a_n must be such that this equation is satisfied for every value of y between 0 and 1. A more manageable equation may be obtained by multiplying Eq. 37 by cos(mπy)dy where m = 1,2,3... and integrating term by term from y = 0 to y = 1. Since

$$\int_0^1 \cos(n\pi y)\cdot\cos(m\pi y)\,dy = \begin{cases} 0, & \text{if } m\neq n \\ \frac{1}{2}, & \text{if } m=n \end{cases} \quad (38)$$

$$\int_0^1 y\cdot\sin(n\pi y)\cdot\cos(m\pi y)\,dy = \begin{cases} (-1)^{m+n}\dfrac{n}{\pi(m^2-n^2)}, & \text{if } m\neq n \\ -\dfrac{1}{4m\pi}, & \text{if } m=n \end{cases} \quad (39)$$

$$\int_0^1 y^2\cos(m\pi y)\,dy = (-1)^m\frac{2}{(m\pi)^2} \quad (40)$$

and

$$\int_0^1 \cos(m\pi y)\,dy = 0 \quad (41)$$

for each value of m, we have

$$\left[K_o(m\pi\delta) + \left(\frac{\pi}{4}\right)^2 \alpha\beta m K_1(m\pi\delta) + \frac{\alpha}{m} K_1(m\pi\delta)\right] a_m$$

$$- 4\alpha \sum_{\substack{n=1 \\ n\neq m}}^{\infty} (-1)^{m+n} \frac{n}{m^2 - n^2} \cdot K_1(n\pi\delta) \cdot a_n = -(-1)^m \frac{2\alpha}{\delta(m\pi)^2} \tag{42}$$

With different values of m, Eq. 42 gives an infinite series of simultaneous linear equations with the infinite number of a_m as unknowns. Since the value of a_m is small for large values of m, an approximate solution to the problem can be obtained by using a finite number of terms under the summation signs in Eq. 35. For this purpose, the values of n and m in Eq. 42 may be limited to a finite value, and the finite number of a_m can be determined from the resultant finite number of simultaneous equations. However, the process of determining the coefficients in this manner is very tedious and the result is only approximate. For the problem under consideration, the value of α in practical cases is extremely small, usually less than 10^{-3}. Thus, as a first approximation, a_m may be computed from

$$\left[K_o(m\pi\delta) + \left(\frac{\pi}{4}\right)^2 \alpha\beta m K_1(m\pi\delta)\right] a_m = -(-1)^m \frac{2\alpha}{\delta(m\pi)^2} \tag{43}$$

A second approximation may be obtained by first computing the values of a_n in Eq. 42 according to Eq. 43:

$$\left[K_o(m\pi\delta) + \left(\frac{\pi}{4}\right)^2 \alpha\beta m K_1(m\pi\delta) + \frac{\alpha}{m} K_1(m\pi\delta)\right] a_m =$$

$$-(-1)^m \frac{2\alpha}{\delta(m\pi)^2} \left\{1 + 4\alpha \sum_{\substack{n=1 \\ n\neq m}}^{\infty} \frac{m^2}{n(m^2 - n^2)\left[\frac{K_o(n\pi\delta)}{K_1(n\pi\delta)} + \left(\frac{\pi}{4}\right)^2 \alpha\beta n\right]}\right\} \tag{44}$$

It can be shown that, if

$$\alpha \frac{K_1(\pi\delta)}{K_o(\pi\delta)} \ll 1 \tag{45}$$

as in nearly all practical cases, Eq. 44 is reduced to Eq. 43: First the term $\frac{\alpha}{m} K_1(m\pi\delta)$ is negligible compared with $K_o(m\pi\delta)$ for any value of m because of the following relationship among the Bessel Functions:

$$\frac{K_1(m\pi\delta)}{K_o(m\pi\delta)} \leq \frac{K_1(\pi\delta)}{K_o(\pi\delta)}, \quad (m \geq 1) \tag{46}$$

Thus

$$\frac{\alpha}{m} K_1(m\pi\delta) \leq \alpha K_1(m\pi\delta) \leq \alpha \frac{K_1(\pi\delta)}{K_0(\pi\delta)} \cdot K_0(m\pi\delta) \ll K_0(m\pi\delta) \quad (47)$$

Secondly, for the same reason, the term involving the summation sign in Eq. 44 is negligible compared with unity:

$$4\alpha \sum_{\substack{n=1 \\ n\neq m}}^{\infty} \frac{m^2}{n(m^2-n^2)\left[\frac{K_0(n\pi\delta)}{K_1(n\pi\delta)} + \left(\frac{\pi}{4}\right)^2 \alpha\beta n\right]} < 4\alpha \sum_{\substack{n=1 \\ n\neq m}}^{\infty} \frac{K_1(n\pi\delta)}{K_0(n\pi\delta)} \cdot \frac{m^2}{n(m^2-n^2)} \quad (48)$$

$$< \alpha \frac{K_1(\pi\delta)}{K_0(\pi\delta)} \cdot 4 \sum_{\substack{n=1 \\ n\neq m}}^{\infty} \frac{m^2}{n(m^2-n^2)} \ll 1$$

Since the condition specified by Eq. 45 is generally satisfied in practice, Eq. 43 is accurate enough for practical purposes and is presented with Eq. 35 as Eq. 9 in the text of this paper.

APPENDIX II - NOTATION

The following symbols, adopted for use in this paper and for the guidance of discussers, conform essentially with American Standard Letter Symbols for Hydraulics (ASA Z10.2 - 1942), prepared by a Committee of the American Standards Association, with Society representation and approved by the Association in 1942:

A = cross sectional area of well;
A_p = area of screen openings per unit area of screen surface;
A_0, A_1, a_0, B_0, B_1, b_0, etc. = constants of integration;
a_m, a_n = constants in solution of Eq. 16;
a, b = constants in well loss formula, Eq. 15;
C_c = coefficient of discharge of screen openings;
$C_s = C_c A_p$, the screen coefficient;
c = constant;
D = inside diameter of well;
D_0 = outside diameter of well;
d = subscript signifying values at discharging end of well;
F, G = functions;
g = gravitational acceleration;
H = piezometric head in aquifer;
h = piezometric head in well;
I_0 = modified Bessel Function of zeroth order of the first kind;
K_0 = modified Bessel Function of zeroth order of the second kind;
K_1 = modified Bessel Function of first order of the second kind;
k = coefficient of permeability of soil (dimensionally length per unit time);
L = thickness of aquifer;

m, n = positive integers;
N_n = function defined in Eq. 9a;
o = subscript signifying values at outside surface of screen;
Q = rate of discharge passing any section of well;
q = discharge per unit cross-sectional area of soil;
r = horizontal distance from center of well;
x = r/L;
y = z/L;
z = vertical distance from bottom of well;
α = $Q_d k D_o/(gA^2)$;

β = $\left(\frac{D}{C_s L}\right)^2$

δ = $D_o/(2L)$;
ϕ = kHL/Q_d

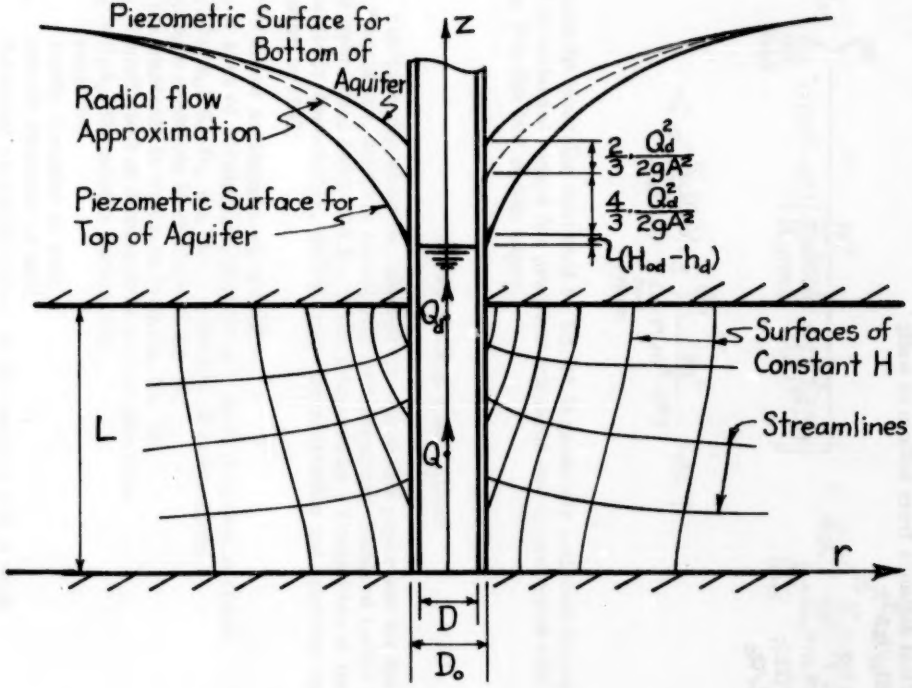

FIG. 1 FLOW INTO A FULLY PENETRATING WELL